of the Parts, and to order in what Direction and how much is neceſſary to be done.

III. *The Continuation of* An Account of a Treatiſe of Fluxions, *&c.* Book II. *by* Colin Mc Laurin, *Prof. Mathem.* Edinburgh. F. R. S.*

Preſented March 10. 1742-3.

IN the firſt Book, the Author deſcribed the Method of Fluxions, and its Application to Problems of different Kinds, without making uſe of any particular Signs or Characters, by geometrical Demonſtrations, that its Evidence might appear in the moſt ſimple and plain Form. In the ſecond Book, he treats of the Method of Computation, or the Algebraic Part; to the Facility, Conciſeneſs, and great Extent of which, the Improvements that have been made by this Method are in great meaſure to be aſcribed. In order to obtain thoſe Advantages, it was neceſſary to admit various Symbols into the Algebra: But the Number and Complication of thoſe Signs muſt occaſion ſome Obſcurity in this Art, unleſs Care be taken to define their Uſe and Import clearly, with the Nature of the ſeveral Operations. An Example of this is given by an Illuſtration of one of the firſt Rules in Algebra. As it is the Nature of Quantity to be capable of Augmentation and Diminution, ſo Addition and Subſtraction are the primary Operations in the Sciences that treat of it. The poſitive Sign implies an Increment,

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* See the Beginning of this Account, N° 468. p. 325.

ment, or a Quantity to be added. The negative Sign implies a Decrement, or Quantity to be ſubſtracted: And theſe ſerve to keep in our View what Elements enter into the Compoſition of Quantities, and in which manner, whether as Increments or Decrements. It is the ſame thing to ſubſtract a Decrement as to add an equal Increment. As the Multiplication of a Quantity by a poſitive Number implies a repeated Addition of the Quantity, ſo the Multiplication by a negative Number implies a repeated Subſtraction: And hence to multiply a negative Quantity, or Decrement, by a negative Number, is to ſubſtract the Decrement as often as there are Units in this Number, and therefore is equivalent to adding the equal Increment the ſame Number of Times; or, when a negative Quantity is multipled by a negative Number, the Product is poſitive. When we inquire into the Proportion of Lines in Geometry, we have no regard to their Poſition or Form; and there is no ground for imagining any other Proportion betwixt a poſitive and negative Quantity in Algebra, or betwixt an Increment and a Decrement, than that of the abſolute Quantities or Numbers themſelves. The Algebraic Expreſſions, however, are chiefly uſeful, as they ſerve to repreſent the Effects of the Operations; and ſuch Expreſſions are not to be ſuppoſed equal that involve equal Quantities, unleſs the Operations denoted by the Signs are the ſame, or have the ſame Effect. Nor is ſuch Expreſſion to be ſuppoſed to repreſent a certain Quantity; for if the $\sqrt{-1}$ ſhould be ſaid to repreſent a certain Quantity, it muſt be allowed to be imaginary, and yet to have a real Square; a way of ſpeaking which it is better to avoid. It denotes only, that an Operation is ſuppoſed to be performed on the Quantity that is under the radical Sign. The Operation is indeed in this Caſe imaginary, or cannot ſucceed; but the Quantity that is under the radical Sign, is not leſs real on that Account. The Author mentions thoſe things briefly, becauſe they belong rather to a Treatiſe of Algebra than of Fluxions, wherein the common Algebra is admitted.

In order to avoid the frequent Repetition of figurative Expreſſions in the Algebraic Part, the Fluxions of Quantities are here defined to be any Meaſures of their reſpective Rates of Increaſe or Decreaſe, while they are ſuppoſed to vary (or flow) together. Theſe may be determined by comparing the Velocities of Points that always deſcribe Lines proportional to the Quantities, as in the Firſt Book; but they may be likewiſe determined, without having recourſe to ſuch Suppoſitions, by a juſt Reaſoning from the ſimultaneous Increments or Decrements themſelves. While the Quantity A increaſes by Differences equal to a, $2A$ increaſes by Differences equal to $2a$, and (ſuppoſing m and n to be invariable) $\frac{mA}{n}$ increaſes by Differences equal to $\frac{ma}{n}$, and therefore at a greater or leſs Rate than a, in proportion as m is greater or leſs than n. Thus a Quantity may be always aſſigned that ſhall increaſe at a greater or leſs Rate than A, (*i. e.* ſhall have its Fluxion greater or leſs than the Fluxion of A) in any Proportion; and a Scale of Fluxions may be eaſily conceived, by which the Fluxions of any other Quantities of the ſame kind may be meaſured.

Let B be any other Quantity whoſe relation to A can be expreſſed by any Algebraic Form; and while A increaſes by equal ſucceſſive Differences, ſuppoſe B to increaſe by Differences that are always varying. In this Caſe, B cannot be ſuppoſed to increaſe at any one conſtant Rate; but it is evident, that if B increaſe by Differences that are always greater than the equal ſucceſſive Differences by which $\frac{mA}{n}$ increaſes at the ſame time, then B cannot be ſaid to increaſe at a leſs Rate than $\frac{mA}{n}$; or if the Fluxion of A be repreſented by a, the Fluxion of B cannot be leſs than $\frac{ma}{n}$. And if the ſucceſſive Differences of B be always leſs than thoſe of $\frac{mA}{n}$, then ſurely B cannot be ſaid to increaſe at a greater Rate than $\frac{mA}{n}$; or the Fluxion of B cannot be ſaid to be greater in this Caſe than $\frac{ma}{n}$.

From thoſe Principles the primary Propoſitions in the Method of Fluxions, and the Rules of the direct Method, with the fundamental Rules of the inverſe Method, are demonſtrated. We muſt be brief in our Account of the Remainder of this Book. The Rule for finding the Fluxion of a Power is not deduced, as uſually, from the Binomial Theorem, but from one that admits of a much eaſier Demonſtration from the firſt Algebraic Elements, *viz.* That when n is any integer poſitive Number, if the Terms E^{n-1}, $E^{n-2}F$, $E^{n-3}F^{2}$, $E^{n-4}F^{3}$, ... F^{n-1},(wherein the Index of E conſtantly decreaſes, and that of F increaſes by the ſame Difference Unit) be multiplied by $E-F$, the Sum of the Products is $E^{n}-F^{n}$; from which it is obvious, that when E is greater than F, then $E^n - F^n$ is leſs than $nE^{n-1} \times \overline{E-F}$ but greater than $nF^{n-1} \times \overline{E-F}$.

The Ruels are ſometimes propoſed in a Form ſomewhat different from the uſual manner of deſcribeing them, with a View to facilitate the Computations both in the direct and inverſe Method. Thus, when a Fraction is propoſed, and the Numerator and Denominator are reſolved into any Factors, it is demonſtrated, that the Fluxion of the Fraction divided by the Fraction is equal to the Sum of the Quotients, when the Fluxion of each Factor of the Numerator is divided by the Factor itſelf, diminiſhed by the Quotients that ariſe by dividing in like manner the Fluxion of each Factor of the Denominator by the Factor.

The Notation of Fluxions is deſcribed in Chap. 2. with the Rules of the direct Method, and the fundamental Rules of the inverſe Method. The latter are comprehended in Seven Propoſitions, Six of which relate to Fluents that are aſſignable in finite Algebraic Terms, and the Seventh to ſuch as are aſſigned by infinite Series. It is in this Place the Author treats of the Binomial and Multinomial Theorems (becauſe of their Uſe on this Occaſion), and they are inveſtigated by the direct Method of Fluxions. The ſame Method is applied for demonſtrating other Theorems, by which an Ordinate of a Figure being given, and its Fluxions determined, any other Ordinate and *Area* of the Figure may be computed. The moſt uſeful Examples are deſcribed in this Chapter, by computing the Series's that ſerve for determining the Arc from its Sine or Tangent, and the

Logarithm from its Number, and converſely the Sine, Tangent, or Secant, from the Arc, and the Number from its Logarithm.

The inverſe Method is proſecuted farther in the Third Chapter, by reducing Fluents to others of a more ſimple Form, when they are not aſſignable by a finite Number of Algebraic Terms. When a Fluent can be aſſigned by the Quadrature of the Conic Sections, (and conſequently by circular Arcs or Logarithms) this is conſidered as the ſecond Degree of Reſolution; and this Subject is treated at Length. An Illuſtration is premiſed of the Analogy betwixt Elliptic and Hyperbolic Sectors formed by Rays drawn from the Centres of the Figures: The Properties of the latter are ſometimes more eaſily diſcovered becauſe of their Relation to Logarithms, and lead us in a brief manner to the analogous Properties of Elliptic Sectors, and particularly to ſome general Theorems concerning the Multiplication and Diviſion of circular Sectors or Arcs. When Two Points are aſſumed in an Hyperbola, and alſo in an Ellipſis, ſo that the Sectors terminated by the Semi-axis; and the Two Semi-diameters, belonging to thoſe Points, are in the ſame given *Ratio* in both Figures, then the Relation betwixt the Semi-axis and the Two Ordinates drawn from thoſe Points to the other Axis, is always defined by the ſame, or by a ſimilar Equation in both Figures. This Propoſition ſerves for demonſtrating Mr. *Cotes*'s celebrated Theorem, as it is extended by M. *De Moivre*, by which a Binomial or Trinomial is reſolved into its quadratic Diviſors, and various Fluents are reduced to circular Arcs and Logarithms. The Demonſtrations are alſo rendred more eaſy of the Theorems concerning the Reſolution of a Fraction, that has a multinomial Denominator, into Fractions that have the ſimple or quadratic Diviſors of the Multinomial for their ſeveral Diviſors. Theſe Demonſtrations are derived from the Method of Fluxions itſelf, without any foreign Aid; or invariable Coefficients are determined by ſuppoſing the variable Quantity or its Fluxions to vaniſh.

When a Fluent cannot be aſſigned by the Arcs of Conic Sections, it may however be meaſured by their Arcs in ſome Caſes; and this may be conſidered as the Third Degree of Reſolution, or the Fluents may be called of the Third Order. On this Occaſion ſome Fluents are found to depend on the Rectification of the Hyperbola and Ellipſis, which have been formerly eſteemed of an higher kind. The Conſtruction of the elaſtic Curve, with its Rectification, and the Meaſure of the Time of Deſcent in an Arch of a Circle, are derived from Hyperbolic and Elliptic Arcs; and the Fluents of this kind are compared with thoſe of the Firſt or Second Order by infinite Series. Becauſe there are Fluents of higher kinds than theſe, the Trajectories above-mentioned, which are deſcribed by a centripetal Force, that is, as ſome Power of the Diſtance from a given Centre, when the Velocity of the Projection is that which would be acquired by an infinite Deſcent, or by ſuch a centrifugal Force, and the Velocity is ſuch as would be acquired by flying from the Centre, are employed for repreſenting them A ſimple Conſtruction of theſe Trajectories had been given above, by drawing Rays from the Centre to a Right Line given in Poſition, increaſing or diminiſhing the Logarithms of thoſe Rays always in a given

Ratio, and increaſing or diminiſhing the Angles contained by them and the Perpendicular in the ſame *Ratio*. From any Figure of this kind a Series of Figures is derived by determining the Interſections of the Tangents of the Figure with the Perpendiculars from the Centre. Every Series of this kind gives Two diſtinct ſort of Fluents; and any one Fluent being given, all the other Fluents taken alternately from it in the Series depend upon it, or are meaſured by it; but it does not appear, that the Fluents of one ſort can be compared with thoſe of the other ſort, or with thoſe of any different Series of this kind.

The inverſe Method is proſecuted farther in the 4th Chapter, by various Theorems concerning the Area when the Ordinate is expreſſed by a Fluent, or when the Ordinate and Baſe are both expreſſed by Fluents. The Firſt is the XIth Propoſition of Sir *Iſaac Newton*'s Treatiſe of Quadratures. In Art. 819, 820, *&c.* the Author ſuppoſes the Ordinate and Baſe to be both expreſſed by Fluents, and ſhews, in many Caſes, that the Area may be aſſigned by the Product of Two ſimple Fluents, as of Two circular Arcs, or of a circular Arc and a Logarithm. This Subject deſerves to be proſecuted, becauſe the Reſolution of Problems is rendered more accurate and ſimple, by reducing Fluents to the Products of Fluents already known, than by having immediately recourſe to infinite Series. One of the Examples in Art. 822. may be eaſily applied for demonſtrating, that the Sum of the Fractions which have Unit for their common Numerator, and the Squares of the Numbers 1, 2, 3, 4, 5, 6, *&c.* in their natural Order, for their ſucceſſive Denominators, is One-ſixth Part of the Number, which expreſſes the *Ratio* of the Square of the Periphery of a Circle to the Square of its Diameter; which is deduced by Mr. *Euler*, *Comment. Petrophol.* Tom. 7. in a different manner; and other Theorems of this kind may be demonſtrated from the ſame or like Principles.

The Series that is deduced by the uſual Methods for computing the Area or Fluent, converge in ſome Caſes at ſo ſlow a Rate, as to be of little or no Uſe without ſome farther Artifice. For Example: The Sum of the firſt Thouſand Terms of Lord *Brounker's* Series for the Logarithm of 2, is deficient in the fifth Decimal. In order therefore to render the Account of the inverſe Method more complete, the Author ſhews how this may be remedied, in many Caſes, by Theorems derived from the Method of Fluxions itſelf, which likewiſe ſerve for approximating readily to the Values of Progreſſions, and for reſolving Problems that are commonly referred to other Methods. Thoſe Theorems had been deſcribed in the Firſt Book, Art. 352, *&c.* but the Demonſtration and Examples were referred to this Place, as requiring a good deal of Computation. The Baſe being ſuppoſed equal to Unit, and its Fluxion alſo equal to Unit, let half the Sum of the extreme Ordinates be repreſented by a, the Difference of the firſt Fluxions of theſe Ordinates by b, the Difference of their Third, Fifth, Seventh and higher alternate Fluxions by c, d, e, *&c.* then the Area ſhall be equal to $a-\frac{b}{12}+\frac{c}{720}-\frac{d}{30240}+\frac{e}{1209600}-$, *&c.* which is the firſt Theorem for finding the Area. The reſt remaining, let *a* now repreſent the middle Ordinate, and the Area ſhall be equal $a + \frac{b}{24} - \frac{7c}{5760} + \frac{31d}{967680} - \frac{127e}{154828800} +$, *&c.* And this is the Theorem which the Author makes moſt Uſe of. When the ſeveral intermediate Ordinates repreſent the Terms of a Progreſſion, the Area is computed from their Sum, or converſely their Sum is derived from the Area, by Theorems that eaſily flow from theſe.

Theſe general Theorems are afterwards applied for finding the Sums of the Powers of any Terms in Arithmetical Progreſſion, whether the Exponents of the Powers be Poſitive or Negative, and for finding the Sums of their Logarithm, and thereby determining the *Ratio* of the *Uncia* of the middle Term of a Binomial of a very high Power to the Sum of all the *Unciæ.* This laſt Problem was celebrated amongſt Mathematicians ſome Years ago, and by endeavouring to reſolve it by the Method of Fluxions the Author found thoſe Theorems, which give the ſame Concluſions that are derived from other Methods. They are likewiſe applied for computing *Areas* nearly from a few equidiſtant Ordinates, and for interpolating the intermediate Terms of a Series, when the Nature of the Figure can be determined, whoſe Ordinates are as the Differences of the Terms.

In the laſt Chapter, the general Rules, derived from the Method of Fluxions for the Reſolution of Problems, are deſcribed and illuſtrated by Examples. After the common Theorems concerning Tangents, the Rules for determining the greateſt and leaſt Ordinates, with the Points of contrary Flexure, and the Precautions that are neceſſary to render them accurate and general, (which were deſcribed above) are again demonſtrated. Next follow the Algebraic Rules for finding the Centre of Curvature, and determining the Cauſtics by Reflexion and Refraction, and the centripetal Forces. The Conſtruction of the Trajectory is given, which is deſcribed by a Force that is inverſely as the Fifth Power of the Diſtance from the Centre, becauſe this Conſtruction requires Hyperbolic and Elliptic Arcs, and becauſe a remarkable Circumſtance takes place in this Caſe, (and indeed in an Infinity of other Caſes) which could not obtain in thoſe that have been already conſtructed by others, *viz.* That a Body may continually deſcend in a ſpiral Line towards the Centre, and yet never approach ſo near to it as to deſcend to a Circle of a certain *Radius*; and a Body may recede for ever from the Centre, and yet never ariſe to a certain finite Altitude. The Conſtruction of the Caſes wherein this obtains is performed by Logarithms or Hyperbolic Areas, the Angles deſcribed about the Centre being always proportional to the Hyperbolic Sectors, while the Diſtances from the Centre are directly or inverſely as the Tangents of the Hyperbola at its Vertex. The Circle is an Aſymptote to the Spiral; and this can never be, unleſs the Velocities requiſite to carry Bodies in Circles increaſe while the Diſtances decreaſe, (or decreaſe while the Diſtances increaſe) in a higher Proportion than the Velocity in the Trajectory; that is, unleſs the Force be inverſely as a higher Power of the Diſtance than the Cube. Next follow Theorems for computing the Time of Deſcent in any Arc of a Curve, for finding the Reſiſtance and Denſity of the Medium when the Trajectory and centripetal

Force are given, and for defining the *Catenaria* and Line of ſwifteſt Deſcent in any Hypotheſis of Gravity.

Then the uſual Rules are derived from the inverſe Method for computing the Area, the Solid generated by it, the Arc of the Curve, and the Surface deſcribed by it revolving about a given Axis. The meridional Parts in a Sphere, and any Spheroid, are determined with the ſame Accuracy, and almoſt equal Facility. The Attraction of a Spheroid at the Equator, as well as at the Poles, is determined in a more general manner than in the Firſt Book, or in a Piece of the Author's publiſhed at *Paris* in 1740. which obtained a Part of the Prize propoſed by the *Royal Academy of Sciences* for that Year. Several Mechanical Problems are reſolved, concerning the Proportion the Power ought to bear to the Weight, that the Engine may produce the greateſt Effect in a given Time; and concerning the moſt advantageous Poſition of a Plane which moves parallel to itſelf, that a Stream of Air or Water may impel it with the greateſt Force, having regard to the Velocity which the Plane may have already acquired. On this Occaſion, it is ſhewn, that the Wind ought to ſtrike the Sails of a Wind-mill in a greater Angle than that of 54° 44′, againſt what has been deduced from the ſame Principles by a learned Author. The ſame Theory is applied to the Motion of Ships, abſtracting from the Lee-way, but having regard to the Velocity of the Ship; and amongſt other Concluſions it appears, that the Velocity of a Veſſel of one Sail may be greater with a Side-wind, than when ſhe ſails directly before the Wind; which, perhaps, may be the Caſe of thoſe ſeen by Captain

Dampier in the *Ladrone Iſlands,* that ſailed at the Rate of 12 Miles in half an Hour with a Side-wind.

The Remainder of this Chapter is employed in reducing Equations from ſecond to firſt Fluxions; conſtructing the elaſtic Curve by the Rectification of the equilateral Hyperbola; determining the Vibrations of Muſical Chords; reſolving Problems concerning the *Maxima* and *Minima,* that are propoſed with Limitations, relating to the Perimeter of the Figure, its Area, the Solid generated by this Area, *&c.* with Examples of this kind concerning the Solid of leaſt Reſiſtance; and concludes with an Inſtance of the Theorems by which the Value of the Ordinate may be determined from the Value of the Area, by common Algebra, and by obſerving, that it is not abſolute, but relative Space and Motion, that is ſuppoſed in the Method of Fluxions.